Multistep Methods

We've been solving IVP's

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

using iterations of the form

$$\begin{cases} w_0 = y_0 \\ w_{n+1} = \boxed{w_n + h f(t_n, w_n)} \end{cases}$$

or $w_{n+1} = \boxed{w_n + h\, T^{(2)}(t_n, w_n)}$

or $w_{n+1} = \boxed{w_n + \frac{h}{2}\left[f(t_n, w_n) + f(t_{n+1}, w_n + h f(t_n, w_n))\right]}$

In all cases, we only used w_n in estimating w_{n+1}

(we never used w_{n-1}, w_{n-2}, etc....)

Multistep methods use several previous estimates to compute the next one.

Idea:

$$y' = f(t,y)$$

$$\Rightarrow \underbrace{\int_{t_n}^{t_{n+1}} y'\,dt}_{y(t_{n+1})-y(t_n)} = \underbrace{\int_{t_n}^{t_{n+1}} f(t,y)\,dt}_{\text{approximate using numerical integration}}$$

Suppose $\int_{t_n}^{t_{n+1}} f(t,y)\,dt$

$$\approx a\,\underbrace{f(t_n,y_n)}_{f_n} + b\,\underbrace{f(t_{n-1},y_{n-1})}_{f_{n-1}}$$

$$+ c\,f(t_{n-2},y_{n-2}) + \cdots$$

or more generally

$$\int_{t_n}^{t_{n+1}} f(t,y)\,dt \approx h\sum_{i=0}^{s-1} a_i f_{n-i}$$

Then, we can write

$$w_{n+1} \approx w_n + h\sum_{i=0}^{s-1} a_i f(t_{n-i}, w_{n-i})$$

↳ Adams-Bashforth Formula of order s

But how do we determine the a_i's?

Undetermined coeff's!
(Take $h=1$, $t_n=0$)

$$\int_0^1 p(t)\,dt = \sum_{i=0}^{s-1} a_i\, p(-i)$$

↳ take
$p(t)=1$
$p(t)=t$
$p(t)=t^2$ or $t(t+1)$
⋮

to get s-equation & s-unknowns (the a_i's)

Example 3:

two-step Adams-Bashforth

$$\boxed{w_{n+1} = w_n + \frac{3}{2} h f(t_n, w_n) - \frac{1}{2} h f(t_{n-1}, w_{n-1})}$$

$(w_0 = y_0)$

$w_1 = \; ?$ → Euler' method (for example)

Exercise: derive this!

Adams-Moulton Formula

Previously, we approximated

$$w_{n+1} = w_n + \underbrace{h \sum_{i=0}^{s-1} a_i f(t_{n-i}, w_{n-i})}$$

"last" term is f

"last" term is f_n

If instead we go all the way up to f_{n+1}:

$$w_{n+1} = w_n + \underbrace{a f_{n+1}} + b f_n + c f_{n-1} \cdots$$

"last" term is f_{n+1}

Then we get an Adams-Moulton formula.

$$w_{n+1} = w_n + h \sum_{i=0}^{s-1} a_i f(t_{n-i+1}, w_{n-i+1})$$

order s

The coefficients a_i can also be determined via undetermined coeff's.

Example AM order 2:

$$w_{n+1} = w_n + \frac{1}{2} h \left[f(t_{n+1}, w_{n+1}) + f(t_n, w_n) \right]$$

Remark:

AM methods have w_{n+1} appear on both sides of the equation so they are called implicit methods since we have to solve an equation (e.g. using Newton's method) to get w_{n+1}.

Alternatively: You could use an A-B method to estimate w^*_{n+1} and use that on the right hand side.

In our example, we could do

$$w^*_{n+1} = w_n + \frac{3}{2} h f(t_n, w_n) - \frac{1}{2} h f(t_{n-1}, w_{n-1})$$

Predictor (AB2)

Corrector (AM2)

$$w_{n+1} = w_n + \frac{h}{2}\left[f(t_{n+1}, w^*_{n+1}) + f(t_n, w_n)\right]$$

where $w_0 = y_0$

w_1 = estimate using some other method (e.g. RK2).

General multi-step methods:

$$\boxed{a_k w_n + a_{k-1} w_{n-1} + \cdots a_0 w_{n-k} = h\left[b_k f_n + b_{k-1} f_{n-1} + \cdots + b_0 f_{n-k}\right]}$$

When $b_k = 0 \to$ Explicit method

$b_k \neq 0 \to$ Implicit method

Want to understand the error

Define :

$$L(y) = \sum_{i=0}^{k} a_i y(ih) - h b_i y'(ih)$$

linear functional

Theorem : The following are equiv.

(1) $L(p) = 0 \quad \forall$ polynomial of p of degree $\leq m$

(2) $L(y)$ is $O(h^{m+1}) \quad \forall y \in C^{m+1}$

we say method is order m

Recall that we derived AB method by satisfying (1), so we get (2) for free!

—x—

Example

What is the order of the method given by:

$$w_n = w_{n-2} + \frac{1}{3}h\left[f_n + 4f_{n-1} + f_{n-2}\right]$$

Let's check

$$\mathcal{L}(y) = y(0) - y(-2h) - \frac{1}{3}h\left[y'(0) + 4y'(-h) + y'(-2h)\right]$$

Check: 1) $y = 1$

$\Rightarrow L(y) = 1 - 1 + \frac{1}{3}[0] = 0$ ✓

2) $y = t$, $y' = 1$

$L(y) = 0 + 2h - \frac{1}{3}h[1 + 4 + 1] = 0$ ✓

3) $y = t^2 \Rightarrow y'(t) = 2t$

$L(y) = 0 - 4h^2 - \frac{1}{3}h\left[0 + 4(-2h) + (-4h)\right]$

4) $y = t^3 \Rightarrow y' = 3t^2$

$L(y) = 0 - (-2h)^3 - \frac{h}{3}\left[0 + 4 \cdot 3(-h)^2 + 1 \cdot 3(-2h)^2\right]$

$= 8h^3 - 4h^3 - 4h^3 = 0$

5) $y = t^4$, ——

$L(y) = \text{——} \neq 0 \Rightarrow O(h^5)$